JOURNAL OF COMPUTATIONAL PHYSICS 32, 10–23 (1979)

Convergence and Optimization of Successive Overrelaxation for Linear Systems of Equations with Complex Eigenvalues

ALAIN RIGAL

Laboratoire de Physique Mathématique, Université Paul Sabatier, 118, route de Narbonne, 31077 Toulouse Cedex, France

Received January 26, 1977; revised October 16, 1978

We apply the point successive overrelaxation method to nonsymmetric linear systems. The matrix of the system is assumed to be consistently ordered but the matrix corresponding to the Jacobi method may have complex eigenvalues. We give the convergence domain of the successive overrelaxation method and an algorithm allowing one to choose the best relaxation factor in this domain.

1. INTRODUCTION

The finite-difference approximation of many physical problems leads to solving large-order sparse linear systems. A number of suitable iterative methods for solving such systems have been studied in detail by many authors [4, 5, 7]. However in most cases, the matrices which have been studied were symmetric and consequently positive definite. In practice, one encounters many problems leading to nonsymmetric linear systems, and the associated Jacobi matrix often has complex eigenvalues.

The point of departure of our analysis is the example of Navier–Stokes equations. They are written here in terms of the streamfunction and the vorticity, and give rise to different finite-difference approximations [1, 3,...].

For nonstationary problems, in the domain $\Omega \times]0, T[$, where Ω is a bounded region of $\mathscr{R}^2$, the partial differential equation can be written as

$$\frac{\partial \zeta}{\partial t} = \frac{1}{R_e} \Delta \zeta - \frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial y}, \tag{1}$$

$$\Delta \psi = -\zeta, \tag{2}$$

u, v given on $\partial\Omega$, at every instant of time,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} : \text{vorticity: curl of the velocity,}$$

$$\frac{\partial \psi}{\partial x} = -v \qquad \frac{\partial \psi}{\partial y} = u : \psi\text{: streamfunction.}$$

0021-9991/79/070010–14$02.00/0

We shall only be concerned with the finite-difference approximation of Eq. (1), which at every point $(i\,\Delta x, j\,\Delta x)$ in the flow domain at the time $n\,\Delta t$ is given by

$$\begin{aligned}(1 + 4a_0)\,\zeta^n_{i,j} - (a_0 + b_0u)\,\zeta^n_{i-1,j} - (a_0 - b_0u)\,\zeta^n_{i+1,j}\\ - (a_0 + b_0v)\,\zeta^n_{i,j-1} - (a_0 - b_0v)\,\zeta^n_{i,j+1} = K.\end{aligned} \tag{3}$$

$$a_0 = \frac{\Delta t}{R_e\,\Delta x^2}, \qquad b_0 = \frac{\Delta t}{2\,\Delta x}, \qquad R_e : \text{Reynolds number},$$

u and v are velocity components which are supposed constant for the study of the matrices.

This equation (3), applied in the entire flow domain, leads to a linear system of which the matrix A is block tridiagonal, the extradiagonal blocks being diagonal. This matrix, associated with a degree-two operator which is elliptic for steady-state problems, has well-known structural properties. With a convenient ordering of the points, the matrix is consistently ordered, which is a sufficient condition for the validity of the fundamental relation (5). In point of fact, all our results will hold for matrices that are such that (5) is true.

To solve this system by a point iterative method, we introduce the Jacobi matrix B and the successive overrelaxation matrix $\mathscr{L}_\omega$ associated with A: $A = D - L - U$, where L is strictly lower triangular, U is strictly upper triangular, and D is diagonal.

$$\begin{aligned}B &= -D^{-1}A + I = D^{-1}(L + U) \qquad \text{Jacobi matrix},\\ \mathscr{L}_\omega &= (D - \omega L)^{-1}\,[(1 - \omega)\,D + \omega U)] \qquad \text{Successive overrelaxation matrix}\\ &\text{for a relaxation factor } \omega.\end{aligned} \tag{4}$$

For matrices A having the structural properties specified above, we have the following fundamental results [5, 7]:

—If μ is any eigenvalue of B of multiplicity p, then $-\mu$ is also an eigenvalue of B of multiplicity p.

—λ, eigenvalue of $\mathscr{L}_\omega$, satisfies (5) for some eigenvalue μ of B; the two roots λ of this equation correspond to $+\mu$ and $-\mu$

$$(\lambda + \omega - 1)^2 = \omega^2\mu^2\lambda. \tag{5}$$

We propose to study one-point successive overrelaxation method, characterized by $\mathscr{L}_\omega$, the spectrum of B being known—or localized in practice—in the complex plane.

We shall therefore study, with the help of Eq. (5), the moduli of the eigenvalues of $\mathscr{L}_\omega$, in terms of μ and ω:

—To render them smaller than unity, in which case, the successive overrelaxation method will be convergent for such values of ω.

—To find a value of ω which minimizes their maximum.

We shall suppose ω different from unity, as in the case of the Gauss–Seidel method; the eigenvalues of this method are the square of those of the Jacobi method.

2. Optimization for the Single-Eigenvalue Problem

Let us first study the single-eigenvalue problem, that is, the convergence and the minimization for $\lambda = x + iy$, given by (5), associated with four eigenvalues $\mu = \pm(\mu_1 \pm i\mu_2)$, for an arbitrary value of ω. We shall get through a different procedure some results already given by Young [7].

Theorem 1. (a) *There exists a value* $\omega_{\max}$ *given by* (11) *such that the successive overrelaxation converges for every* ω *in the interval* $]0, \omega_{\max}[$, *if and only if* $\mu \in \mathscr{D} =]-1, +1[\times \mathscr{R}$, *domain of the complex plane.*

(b) *There exists, in this convergence interval, only one optimum value of* ω *which maximizes the rate of convergence of the successive overrelaxation and which is given by* (16), (17).

Proof. (a) (5) leads to two real equations:

$$\begin{aligned} x^2 - y^2 - \omega(\mu_1 x - \mu_2 y) + \omega - 1 &= 0, \\ 2xy - \omega(\mu_1 y + \mu_2 x) &= 0. \end{aligned} \tag{6}$$

We obtain from (6) a polynomial equation in terms of $R = x^2 + y^2$.

$$\begin{aligned} R^4 - \omega^2(\mu_1^2 + \mu_2^2)\, R^3 - 2(1-\omega)[(1-\omega) + \omega^2(\mu_1^2 - \mu_2^2)]\, R^2 - \omega^2(1-\omega)^2 \\ \times (\mu_1^2 + \mu_2^2)\, R + (1-\omega)^4 = 0 \end{aligned} \tag{7}$$

We let $R = (1 - \omega)\, S$ which makes Eq. (7) symmetric; then, with $S + 1/S = T$, we obtain a second-degree equation with respect to T.

$$\begin{aligned} (1-\omega)\, T^2 - \omega^2(\mu_1^2 + \mu_2^2)\, T - 2[2 - 2\omega + \omega^2(\mu_1^2 - \mu_2^2)] &= 0, \\ R^2 - RT(1-\omega) + (1-\omega)^2 &= 0. \end{aligned} \tag{8}$$

Taking the only convenient solution, we obtain

$$\begin{aligned} R &= \frac{T_0 + [T_0^2 - 16(1-\omega)^2]^{1/2}}{4}, \\ T_0 &= \omega^2(\mu_1^2 + \mu_2^2) + [\omega^4(\mu_1^2 + \mu_2^2)^2 + 8\omega^2(1-\omega)(\mu_1^2 - \mu_2^2) + 16(1-\omega)^2]^{1/2}. \end{aligned} \tag{9}$$

From this, it appears that, for every ω and every μ, (7) has only two real roots of which the product is $(1 - \omega)^2 < 1$. The convergence condition $R < 1$ will be $P(1) > 0$, $P(R)$ being the polynomial equation (7)

$$\omega^2(1 - \mu_1^2 - \mu_2^2) + 4(\omega - 1)(\mu_1^2 - 1) > 0. \tag{10}$$

Thus the SOR method will converge for every value of ω in the interval $]0, \omega_{\max}[$ with

$$\omega_{\max} = \frac{2}{1 + |\mu_2|/(1 - \mu_1^2)^{1/2}} \tag{11}$$

which exists if and only if $\mu \in \mathscr{D} =]-1, +1[\times \mathscr{R}$. Provided that ω satisfies this unequality, the convergence domain of the point overrelaxation method is $\mathscr{D}$, whereas that of the Jacobi and Gauss–Seidel methods is the unit disk.

(b) Inside this interval, we seek the value of ω, if it exists, which minimizes R. We differentiate (9) with respect to ω and set this differential to zero. After a series of tedious calculations one obtains

$$A_0\omega^6 + A_1\omega^4(1 - \omega) + A_2\omega^2(1 - \omega)^2 + A_3(1 - \omega)^3 = 0 \tag{12}$$

with

$$A_0 = A^4 - A^2B, \qquad A = \mu_1^2 + \mu_2^2,$$
$$A_1 = 12A^2B - 3A^2 - 9B^2, \qquad B = \mu_1^2 - \mu_2^2,$$
$$A_2 = 24A^2 + 24B^2 - 48B,$$
$$A_3 = 16A^2 - 16B^2 + 64B - 64$$

from which ensues the third-degree polynomial equation with respect to $\Omega = \omega^2 \times (1 - \omega)^{-1}$

$$A_0\Omega^3 + A_1\Omega^2 + A_2\Omega + A_3 = 0. \tag{13}$$

The roots of (13) are obtained by the classical procedure

$$\Omega_1 = \Omega + \frac{A_1}{3A_0} \Rightarrow \Omega_1^3 + p\Omega_1 + q = 0 \tag{14}$$

with

$$p = \frac{3a(8b^2 - a)}{A_0^2}, \qquad a = A^2 - B^2 = 4\mu_1^2\mu_2^2,$$

$$q = \frac{2a(8b^4 - 20ab^2 - a^2)}{A_0^3}, \qquad b = A^2 - B = (\mu_1^2 + \mu_2^2)^2 - (\mu_1^2 - \mu_2^2),$$

$$\Delta = \frac{p^3}{27} + \frac{q^2}{4} = \frac{64a^2b^2(a + b^2)^3}{A_0^6} > 0 \qquad \forall \mu_1, \mu_2. \tag{15}$$

For every value of μ_1, μ_2, we have only one real root in terms of Ω_1, and thus in terms of Ω, given by (16)

$$\Omega = \frac{1}{A^4 - A^2B}\{[3b + (a + b^2)^{1/2}]\, a^{1/3}[(a + b^2)^{1/2} - b]^{1/3}$$
$$- [3b - (a + b^2)^{1/2}]\, a^{1/3}[(a + b^2)^{1/2} + b]^{1/3} + A^2 + 3B^2 - 4A^2B\}. \tag{16}$$

We can thus obtain ω from $\omega^2 - (\omega - 1)\,\Omega = 0$. It is rather easy to verify that there is only one ω optimum, for every μ belonging to $\mathscr{D}$, given by

$$\omega_{\text{opt}} = -\frac{\Omega - (\Omega^2 + 4\Omega)^{1/2}}{2} \qquad \text{if } A^2 - B > 0 \text{ which corresponds to } \omega_{\text{opt}} < 1,$$

$$\omega_{\text{opt}} = -\frac{\Omega + (\Omega^2 + 4\Omega)^{1/2}}{2} \qquad \text{if } A^2 - B < 0 \text{ which corresponds to } \omega_{\text{opt}} > 1. \tag{17}$$

The case where $A^2 - B$ is equal to zero does not occur here; it corresponds to $\omega_{\text{opt}} = 1$. Thus we have $(\mu_1^2 + \mu_2^2)^2 = \mu_1^2 - \mu_2^2$, that is, $\mu \in (\mathscr{L})$, $(\mathscr{L})$ a Bernoulli lemniscate in the plane (μ_1, μ_2) [2].

3. Optimization for the Two-Eigenvalue Problem

We now propose to study the general problem of the minimization of $[R(\omega, \mu_1, \mu_2)]$ $\mu \in \Sigma$, with Σ a given arbitrary spectrum contained within $\mathscr{D}$, for linear systems having Young's properties. The interval of variation of ω will be $]0, \omega_m[$ with

$$\omega_m = \frac{2}{1 + b_m}, \qquad b_m = \max_{\mu \in \Sigma} \frac{|\mu_2|}{(1 - \mu_1^2)^{1/2}}, \tag{18}$$

$(1, b_m)$ are the real and imaginary semi-axes of the ellipse containing all elements of Σ. The existence of ω optimum minimizing $R(\omega, \mu)$, for every μ inside $\mathscr{D}$, only allows us to affirm that every curve among $[R(\omega, \mu)]_{\mu \in \Sigma}$ has a minimum with respect to ω, but not to foresee the existence of an extremal $R(\omega, \mu^*)$—for the spectral radius μ^* of B, in the real case. By expanding (9) in the neighborhood of $\omega = 0$, we can establish the properties of these curves in this neighborhood

$$R'_\omega(0, \mu_1, \mu_2) = \mu_1 - 1. \tag{19}$$

First we study the minimization problem for a two-eigenvalue spectrum $\Sigma = (\mu, \mu + d\mu)$: $\min_\omega [R(\omega, \mu), R(\omega, \mu + d\mu)]$ can occur for $\min_\omega R(\omega, \mu)$, $\min_\omega R(\omega, \mu + d\mu)$, or $R(\omega, \mu) = R(\omega, \mu + d\mu)$ if this intersection exists.

$R(\omega, \mu)$ and $R(\omega, \mu + d\mu)$ will have an intersection if

$$R'_{\mu_1}\, d\mu_1 + R'_{\mu_2}\, d\mu_2 = 0 \tag{20}$$

has a solution in terms of ω : $\omega_I \in]0, \omega_m[$. We obtain this solution by differentiating relations (8). Thus we have

$$\omega_I = \frac{2}{1 + (1 - (a_1^2 - b_1^2))^{1/2}} \tag{21}$$

with

$$a_1^2 = \frac{(\mu_2 + d\mu_2)^2 \mu_1^2 - \mu_2^2(\mu_1 + d\mu_1)^2}{(\mu_2 + d\mu_2)^2 - \mu_2^2},$$

$$b_1^2 = \frac{(\mu_1 + d\mu_1)^2 \mu_2^2 - \mu_1^2(\mu_2 + d\mu_2)^2}{(\mu_1 + d\mu_1)^2 - \mu_1^2},$$

a_1 and b_1 being the semi-axes of the ellipse which passes through μ and $\mu + d\mu$ in the plane (μ_1, μ_2); ω_I exists if

$$d\mu_1\, d\mu_2 < 0 \qquad \text{and} \qquad a_1^2 - b_1^2 < 1.$$

In that case, R takes the value [5]

$$R(\omega_I, \mu) = (\omega_I - 1)\frac{a_1 + b_1}{a_1 - b_1}. \tag{22}$$

We can conclude that, if ω_I exists, it belongs to $]0, \omega_m[$. The problem of the minimization of R for $(\mu, \mu + d\mu)$ will be treated in the following way:

—ω_I does not exist. Because of (19), it is for the eigenvalue of which the real part is the greatest that one must minimize R,

$$\omega_{\text{opt}}(\Sigma) = \begin{cases} \omega_{\text{opt}}(\mu) & \text{if} \quad d\mu_1 < 0, \\ \omega_{\text{opt}}(\mu + d\mu) & \text{if} \quad d\mu_1 > 0. \end{cases} \tag{23}$$

—If ω_I exists, we have to place it with respect to $\omega_{\text{opt}}(\mu)$ and $\omega_{\text{opt}}(\mu + d\mu)$; we shall suppose $\omega_{\text{opt}}(\mu) < \omega_{\text{opt}}(\mu + d\mu)$ without loss of generality. We will have three cases

$$\omega_I \in]\omega_{\text{opt}}(\mu), \omega_{\text{opt}}(\mu + d\mu)[\Rightarrow \omega_{\text{opt}}(\Sigma) = \omega_I \Leftrightarrow \begin{cases} R'_\omega(\omega_I, \mu) > 0 \\ R'_\omega(\omega_I, \mu + d\mu) < 0 \end{cases} \tag{24.1}$$

$$\omega_I \in]0, \omega_{\text{opt}}(\mu)[\Rightarrow \omega_{\text{opt}}(\Sigma) = \begin{cases} \omega_{\text{opt}}(\mu) & \text{if} \quad d\mu_1 > 0 \\ \omega_{\text{opt}}(\mu + d\mu) & \text{if} \quad d\mu_1 < 0 \end{cases} \overset{\Leftrightarrow}{} \begin{cases} R'_\omega(\omega_I, \mu) < 0 \\ R'_\omega(\omega_I, \mu + d\mu) < 0 \end{cases} \tag{24.2}$$

$$\omega_I \in]\omega_{\text{opt}}(\mu + d\mu), \omega_m[\Rightarrow \omega_{\text{opt}}(\Sigma) = \begin{cases} \omega_{\text{opt}}(\mu) & \text{if } d\mu_1 < 0 \\ \omega_{\text{opt}}(\mu + d\mu) & \text{if } d\mu_1 > 0 \end{cases} \overset{\Leftrightarrow}{} \begin{cases} R'_\omega(\omega_I, \mu) > 0 \\ R'_\omega(\omega_I, \mu + d\mu) > 0 \end{cases} \tag{24.3}$$

The determination of ω optimum by means of (16) and (17) being rather long, we have, for the purpose of the generalization of the process of the search of ω optimum, merely translated the right-hand inequalities. To do this, we differentiate (9) and set $\omega = \omega_I$;

$$\frac{a_1 b_1}{(a_1^2 + b_1^2) - (\mu_1^2 + \mu_2^2)} < \frac{1}{(1 - (a_1^2 - b_1^2))^{1/2}} \Leftrightarrow R'_\omega(\omega_I, \mu_1, \mu_2) < 0 \quad \text{if} \quad \omega_I < 1. \tag{25}$$

The inequality is inverted if $\omega_l > 1$. This test (25) requires only the knowledge of the values $(\mu, \mu + d\mu)$; it allows us to know quickly in which case (24.1, 2, 3) we are in, and to deduce $\omega_{\text{opt}}(\Sigma)$, given by (16), (17), or by (21) according to the case: It is above all very useful for obtaining a generalization.

4. General Algorithm

With the results collected in the preceding paragraphs, we can construct an algorithm for the search of ω_{opt} which maximizes the rate of convergence of the point overrelaxation method. The fundamental equation (5) allows us to study the spectrum $\Sigma = \{|\mu_1| + i|\mu_2|\}$, i.e., to transfer the initial spectrum in the first quadrant of the μ-plane without loss of generality. We shall sweep the interval $]0, \omega_m[$ and frequently use the test (25). The following constitutes the principal stages of the algorithm.

[A] First one determines for a given spectrum (Σ), the convergence domain, that is, the ellipse with semi-axes 1 and b_m containing (Σ), hence

$$\omega_m = \frac{2}{1 + b_m}, \qquad b_m = \max_{\mu \in \Sigma} \frac{|\mu_2|}{(1 - \mu_1^2)^{1/2}}.$$

[B] One spans $]0, \omega_m[$ in the following way:

—B1: One searches for an eigenvalue μ^0 belonging to Σ such that

$$\mu_1^0 = \sup_{\mu \in \Sigma} \mu_1^i,$$

because of relation (19).

—B2: One is led to the study of the spectrum

$$\Sigma_1 = \{\mu^i : \mu_2^i > \mu_2^0\}_{i \in I_1} \tag{26}$$

because of relation (23)

—B3: One calculates the intersections

$$\{\omega^{i_0} : R(\omega, \mu^0) = R(\omega, \mu^i)\}_{i \in I_1}$$

with μ^1 such that $\omega^{10} = \inf_{i \in I_1} \omega^{i_0}$. If ω^{i_0} doesn't exist, for any i, $\omega_{\text{opt}}(\Sigma) = \omega_{\text{opt}}(\mu^0)$

—B4: One studies the signs of $R'_\omega(\omega^{10}, \mu^0)$ and $R'_\omega(\omega^{10}, \mu^1)$ by means of test (25).

[C] Four cases can occur:

—C1: $R'_\omega(\omega^{10}, \mu^0) < 0$ and $R'_\omega(\omega^{10}, \mu^1) < 0 \rightarrow$ (24.2) : in which case one goes on sweeping at stage B2 by restriction to the study of the spectrum

$$\Sigma_2 = \{\mu^i : \mu_2^i > \mu_2^1\}_{i \in I_2}$$

One calculates the intersections

$$\{\omega^{i_1}\colon R(\omega, \mu^1) = R(\omega, \mu^i)\}_{i \in I_2} \tag{27}$$

if ω^{i_1} doesn't exist, for any i, $\omega_{\text{opt}}(\Sigma) = \omega_{\text{opt}}(\mu^1)$, otherwise, one searches $\omega^{21} = \inf_{i \in I_2}(\omega^{i_1})$, which is submitted to the test in B4 and one goes on to C

—C2: $R'_\omega(\omega^{10}, \mu^0) > 0$ and $R'_\omega(\omega^{10}, \mu^1) > 0 \rightarrow$ (24.3)

Then we have $\omega_{\text{opt}}(\Sigma) = \omega_{\text{opt}}(\mu^0)$

—C3: $R'_\omega(\omega^{10}, \mu^0) < 0$ and $R'_\omega(\omega^{10}, \mu^1) > 0 \rightarrow$ (24.1)

Then we have $\omega_{\text{opt}}(\Sigma) = \omega^{10}$

—C4: $R'_\omega(\omega^{10}, \mu^0) > 0$ and $R'_\omega(\omega^{10}, \mu^1) < 0$

This case cannot occur because we have choosen μ^0 such that $R(\omega, \mu^0)$ is extremal when ω is equal to zero. Evidently, if there exists μ belonging to Σ such that

$$\begin{matrix} \mu_1 \geqslant \mu_1{}^i \\ \\ \mu_2 \geqslant \mu_2{}^i \end{matrix} \qquad \forall i \in I.$$

One applies Theorem 1 for this eigenvalue.

THEOREM 2. (a) *For every linear system having Young's properties, and such that the spectrum of the associated Jacobi matrix is contained in* $\mathscr{D} =]-1, +1[\times \mathscr{R}$ *there exists an interval* $]0, \omega_m[= J$ *determined by* (18), *such that the point overrelaxation method converges for every value* ω *belonging to* J.

(b) *If all of the eigenvalues of the Jacobi matrix are known, then by means of Theorem* 1 *and relations* (19), (21), (25), *one can construct an algorithm* (26), (27) *for the determination of the value of* ω_{opt}, *belonging to* J, *which will maximize the rate of convergence of the successive overrelaxation method.*

5. NUMERICAL EXPERIMENTATIONS

We have not made a complete review of the numerical applications of Theorems 1 and 2. To deduce practical important information, we have set the numerical studies in three directions.

5.1. *Single-Eigenvalue Problem*

For the single-eigenvalue problem, i.e., to apply Theorem 1 and to bring out the behavior of Eqs. (16) and (17), we have drawn Figs. 1 and 2. They show the respective influence of the real and imaginary parts of the Jacobi eigenvalue in the minimization of the spectral radius of $\mathscr{L}_\omega$.

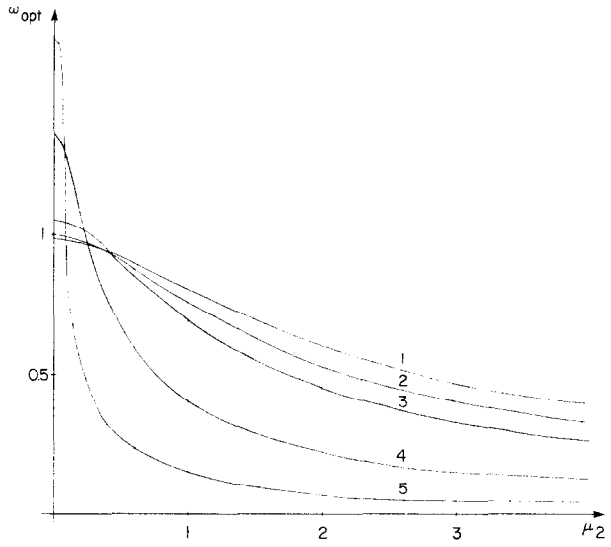

FIG. 1. Variations of ω_{opt} depending of μ_2 for (1) $\mu_1 = 0$, (2) $\mu_1 = 0.2$, (3) $\mu_1 = 0.5$, (4) $\mu_1 = 0.9$, (5) $\mu_1 = 0.99$.

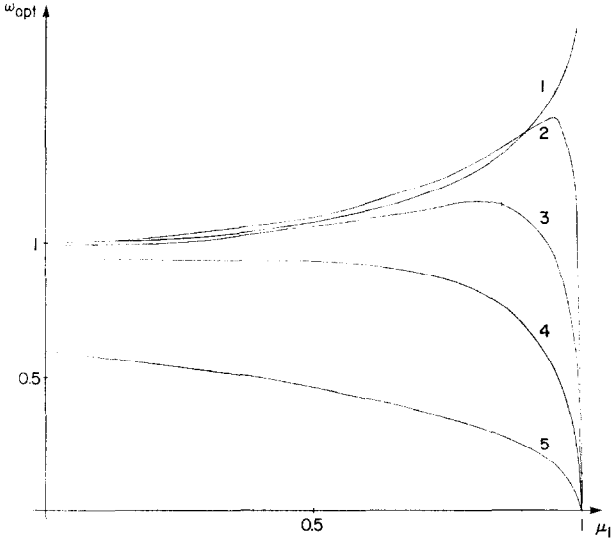

FIG. 2. Variations of ω_{opt} depending of μ_1 for (1) $\mu_2 = 0$, (2) $\mu_2 = 0.05$, (3) $\mu_2 = 0.2$, (4) $\mu_2 = 0.5$, (5) $\mu_2 = 2$.

In Fig. 1, we can see that

$$\omega_{opt} = \frac{2}{1 + (1 - \mu_1^2)^{1/2}}$$

when $\mu_2 = 0$, and decreases to 0 when μ_2 increases, the faster μ_1 tends to 1.

Fig. 2 shows us that

$$\omega_{\text{opt}} = \frac{2}{1 + (1 + \mu_2^2)^{1/2}} < 1$$

when $\mu_1 = 0$ and is strictly decreasing only if $\mu_2 > 0.5$.

Note that a discontinuity appears in ω_{opt} for $\mu_2 = 0$: $\mu_2 = 0$, $\lim_{\mu_1 \to 1} \omega_{\text{opt}} = 2$; $\mu_2 \neq 0$, $\lim_{\mu_1 \to 1} \omega_{\text{opt}} = 0$.

Thus, we have a good idea of the behavior of ω_{opt} depending on the values of μ_1 and μ_2. These remarks and graphs are important because there are many practical problems where we find

$$\mu^0 = \mu_1^0 + i\mu_2^0 \in \Sigma$$

such that:

$$\mu_1^0 \geqslant \mu_1 \qquad \text{and} \qquad \mu_2^0 \geqslant \mu_2 \qquad \forall \mu \in \Sigma$$

In which case we apply Eqs. (16) and (17) to μ^0.

5.2 *Multiple-Eigenvalue Problem*

We have applied the algorithm (26), (27) for arbitrary spectrums. The program utilized is exactly the transposition of the different steps of the algorithm and is very significantly shorter and faster than the program described in [8]; we can hence widely increase the number of Jacobi eigenvalues without any difficulty—10^3 or more... Among several numerical examples, we retain two test problems that allow us to explain and visualize the execution of our algorithm and to draw some fundamental remarks. First, we have considered the model spectrum studied in [8]: it has fifty

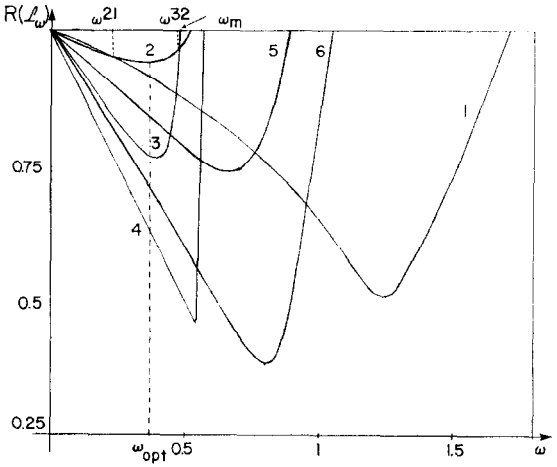

FIG. 3. Variations of $R(\mathscr{L}_\omega)$ for the first model spectrum: The number of the curves are the index of the eigenvalues.

arbitrary eigenvalues. One can see in Fig. 3, the variations of $R(\mathscr{L}_\omega)$ with respect to ω, for six main eigenvalues of this spectrum.

$$\lambda_1 = 0.8 + 0.1i, \qquad \lambda_3 = 0.3 + 3i, \qquad \lambda_5 = 0.6 + i,$$
$$\lambda_2 = 0.78654 + 1.75432i, \qquad \lambda_4 = 2.5i, \qquad \lambda_1 = 0.25 + 0.875i.$$

We describe in this case the execution of our algorithm, as we see it in Fig. 3:

—Our algorithm begins in [A] with the computation of $\omega_m(\Sigma)$,

$$\omega_m(\Sigma) = \omega_m(\lambda_3) = 0.4825.$$

—Then, in [B1], we retain λ_1 such that $\mathrm{Re}(\lambda_1) \geqslant \mathrm{Re}(\lambda_i)$, $1 \leqslant i \leqslant 50$.

— [B2] keeps only the eigenvalues such that $\mathrm{Im}(\lambda_i) > \mathrm{Im}(\lambda_1) : (\Sigma_1)_{I_1}$

— [B3] calculates the values of the intersections ω^{i1} and selects

$$\omega^{21} = \inf_{i \in I_1} (\omega^{i1}) = 0.1885$$

— [B4] gives the signs of $R'_\omega(\omega^{21}, \lambda_1)$ and $R'_\omega(\omega^{21}, \lambda_2)$ which are < 0 : case [C1].

—Thus we go on in [B2] with $\Sigma_2 = \{\lambda_3, \lambda_4\}$ because

$$\mathrm{Im}(\lambda_3), \mathrm{Im}(\lambda_4) > \mathrm{Im}(\lambda_2).$$

— [B3] gives only one intersection $\omega^{32} = 0.4784$.

— [B4] shows us that $R'_\omega(\omega^{32}, \lambda_2)$ and $R'_\omega(\omega^{32}, \lambda_3)$ are > 0.

Hence we conclude by [C2] that

$$\omega_{\mathrm{opt}}(\Sigma) = \omega_{\mathrm{opt}}(\lambda_2) = 0.3574.$$

Then we examine the following model spectrum

$$\Sigma = \{\mu^k = 0.009k + i(2 - 0.02k)\}_{1 \leqslant k \leqslant 100}$$

In Fig. 4, we describe the variations of $R(\mathscr{L}_\omega)$ for seven significant eigenvalues of Σ : $\mu^1, \mu^{20}, \mu^{40}, \mu^{60}, \mu^{80}, \mu^{90}, \mu^{100}$. On this set of curves, we can easily see that:

$\omega_{\mathrm{opt}}(\Sigma) = \omega_I = 0.6601$ corresponding to the intersection of R_{μ^1} and $R_{\mu^{100}}$, this result obtained through our algorithm is very close to $\omega_m(\Sigma) = \omega_m(\mu^1) = 0.673$.

The influence of the real and imaginary parts of the Jacobi eigenvalues and the direct applications of our results shall are clear from Figs. 1, 2, 3, 4; we shall only bring attention to the following fundamental point:

For a complex eigenvalue or spectrum problem, if we can only obtain an approximate value of ω_{opt} (see Section 5.3) we must overestimate it if the imaginary parts of the eigenvalues are weak, and underestimate it if they are large. In the second case, that is absolutely necessary to avoid divergence of the successive overrelaxation.

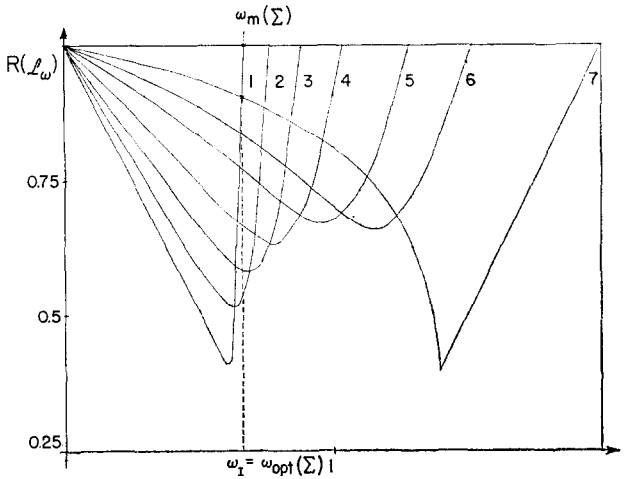

FIG. 4. Variations of $R(\mathcal{L}_\omega)$ for the second model spectrum: the number of the curves are for (1) μ_1, (2) μ^{20}, (3) μ^{40}, (4) μ^{60}, (5) μ^{80}, (6), μ^{90}, (7) μ^{100}.

5.3 *Navier–Stokes Equations*

This work originally rose from the Navier–Stokes equations, i.e., the linear system obtained from the finite difference equation (3). The coefficients u, v are variable, so we cannot apply this study directly, but it is possible to deduce from Theorems 1 and 2 some very important information to solve the linear system (3).

To bring out that, we have compared for a model problem in a rectangle the experimental value of ω_{opt} obtained by solving the linear system (3) and ω_{opt} deduced from Theorems 1 and 2, with u and v supposed constants. With these assumptions we know the Jacobi eigenvalues associated to the system (3), which are generally complex—U and/or V greater than unity.

$$\mu_{pq} = \frac{2a_0}{1 + 4a_0}\left[(1 - U^2)^{1/2} \cos\frac{p\pi}{l} + (1 - V^2)^{1/2} \cos\frac{q\pi}{m}\right] \tag{28}$$

with

$$U = -\frac{b_0}{a_0}u, \qquad V = \frac{b_0}{a_0}v, \qquad p = 1,\ldots, l-1, \quad q = 1,\ldots, m-1,$$

where l, m are the number of grid-points in the two directions and a_0 and b_0 are given by (3).

Table I shows us the values of ω_{opt} obtained by applying Theorems 1 and 2 to the spectrum defined by (28), taking—or estimating—U_{min}, U_{max}, V_{min}, V_{max} in the domain and for several values of the Reynolds number.

This table must be compared with the experimental values of ω_{opt} in Fig. 5: We give the iterations number for the effective resolution of the system when ω varies between 0 and ω_m. Table I gives satisfactory values of ω_{opt} as compared with the effective

TABLE I

Values of ω_{opt} for Several Reynolds Numbers

Re	0.1	1	10	20	50	100	400
ω_{opt}	1.51	1.39	1.01	0.92	0.81	0.78	0.75

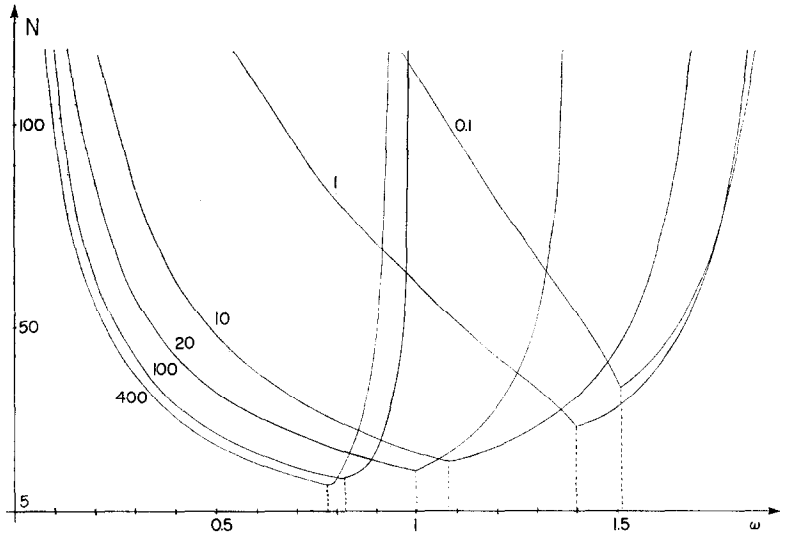

FIG. 5. N: Number of iterations depending of ω for several Reynolds numbers, showing the effective experimental value of ω_{opt}.

values shown in Fig. 5. In particular, ω_{opt} is underestimated when R_e is large, which avoids possible divergence.

6. CONCLUSION

This algorithm has the same purpose as the one given by Young to solve the same problem [7]. The Young algorithm utilizes many more geometrical considerations, whereas our algorithm is strictly analytical and does not necessitate any auxiliary numerical solution. However like the Young algorithm, it assumes theoretically the knowledge of all the eigenvalues of the spectrum Σ, which is a very great demand for large-order systems.

Evidently, most eigenvalues do not have any influence. We must therefore determine the most important eigenvalues, or at least localize them. We can do this for physical problems, when the coefficients are constants, or estimate them when they are variable as for the Navier–Stokes equations.

It is quite evident that the remaining difficulty for an arbitrary linear system is to know the Jacobi spectrum or to obtain the maximum informations on this spectrum.

REFERENCES

1. G. ALEIX, Thèse de Doctorat de Spécialité, Université Paul Sabatier, Toulouse, 1972.
2. G. KJELLBERG, *Ericsson Technics Stockholm* **2** (1958), 245–258.
3. D. B. RUSSEL, "On Obtaining Solutions to the Navier–Stokes Equations with Automatic Digital Computers," Aeronautical Research Council Report R & M 3331, Engineering Laboratory, Oxford, 1963.
4. R. S. VARGA, "Matrix Iterative Analysis," Prentice–Hall, Englewood Cliffs, N. J., 1962.
5. E. L. WACHSPRESS, "Iterative Solution of Elliptic Systems and Applications to the Neutron Diffusion Equations of Reactor Physics," Prentice–Hall, Englewood Cliffs, N. J., 1966.
6. E. E. WRIGLEY, *Comput. J.* (1963), 169–176.
7. D. M. YOUNG, "Iterative Solution of Large Linear Systems," Academic Press, New York, 1971.
8. D. M. YOUNG AND H. E. EIDSON, "On the Determination of the Optimum Relaxation Factor for the SOR Method when the Eigenvalues of the Jacobi Method are Complex," Report CNA-1, Center for Numerical Analysis, Univ. of Texas, Austin, 1970.